Polyhedral geometry of bisectors and bisection fans

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(Joint work with Katharina Jochemko)

Polyhedral norms

- Let $P \subset \mathbb{R}^d$ be a centrally symmetric, convex body with $0 \in \text{int}(P)$ and let its associated *distance function* be

$$\text{dist}^P(a, b) = \min\{\alpha \in \mathbb{R}_{\geq 0} : b - a \in \alpha P\}.$$

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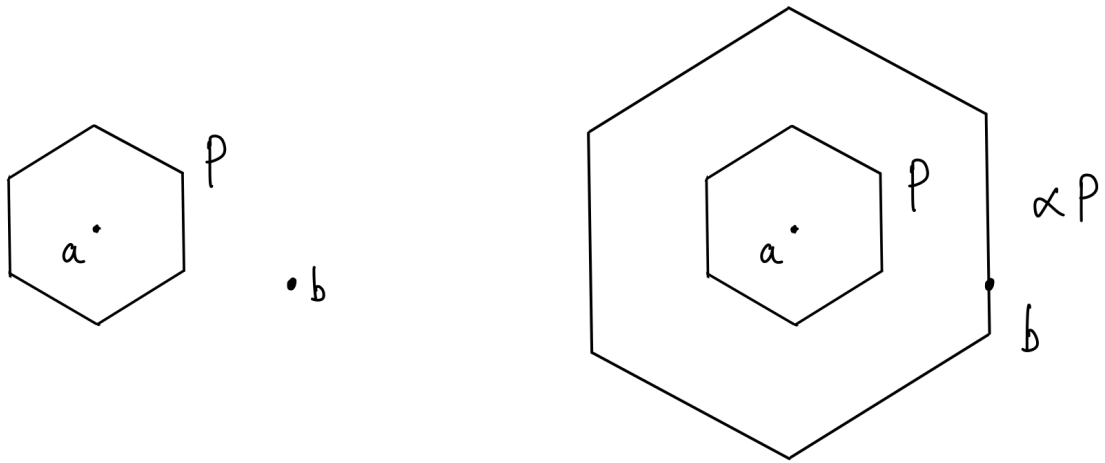


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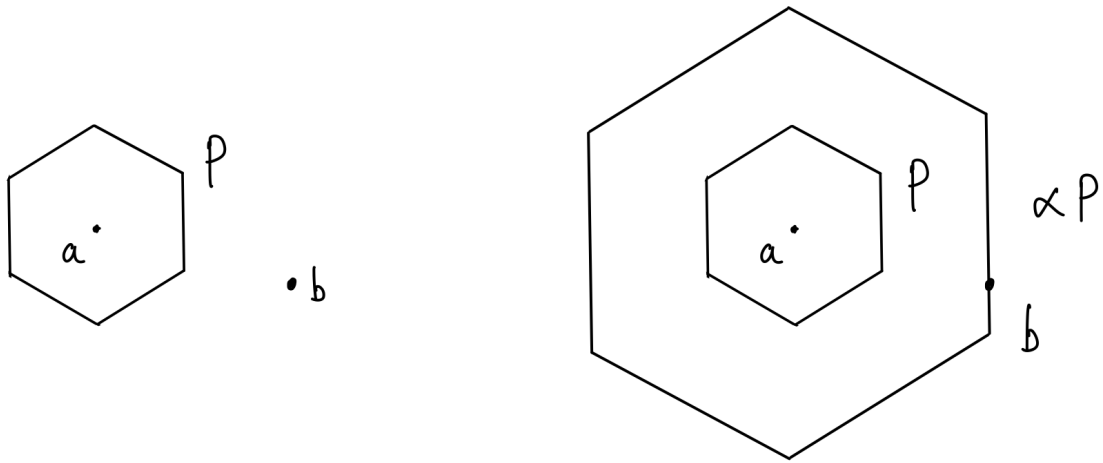


Figure: For the hexagon P, the distance between a and b is $\text{dist}^P(a, b) = \alpha$.

- When P is a polytope, the function $\text{dist}^P(0, \cdot)$ is called a *polyhedral norm* on $\mathbb{R}^d$.

Properties of dist

1. **(Non-negativity)** $\text{dist}^P(0, x) \geq 0$.

2. **(Homogeneity)** $\text{dist}^P(0, \lambda x) = \lambda \, \text{dist}^P(0, x)$ for $\lambda \geq 0$.

3. **(Sub-additivity)** $\text{dist}^P(0, x + y) \leq \text{dist}^P(0, x) + \text{dist}^P(0, y)$.

- **(Piecewise linear)** For a centrally symmetric polytope P, $\text{dist}^P(0, \cdot)$ is piecewise linear with regions of linearity equal to sets of the form C_F, the cone over the facet F of P.

dist in different guises

- Centrally symmetric convex bodies in $\mathbb{R}^d$ are in 1-to-1 correspondence with polyhedral norms on $\mathbb{R}^d$:

$$K \longmapsto \text{dist}^K(0, \cdot)$$
$$f^{-1}([0,1]) \longmapsfrom f : \mathbb{R}^d \to \mathbb{R}$$

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- Distance functions in other contexts:

- As *gauge functions* in the geometry of numbers and toric geometry.

- As *radial functions* in the study of convex and star-convex sets.

Bisectors

- Given two points $a, b \in \mathbb{R}^d$, consider the *bisector* of a and b:

$$\text{bis}^P(a, b) = \{x \in \mathbb{R}^d : \text{dist}^P(a, x) = \text{dist}^P(b, x)\}.$$

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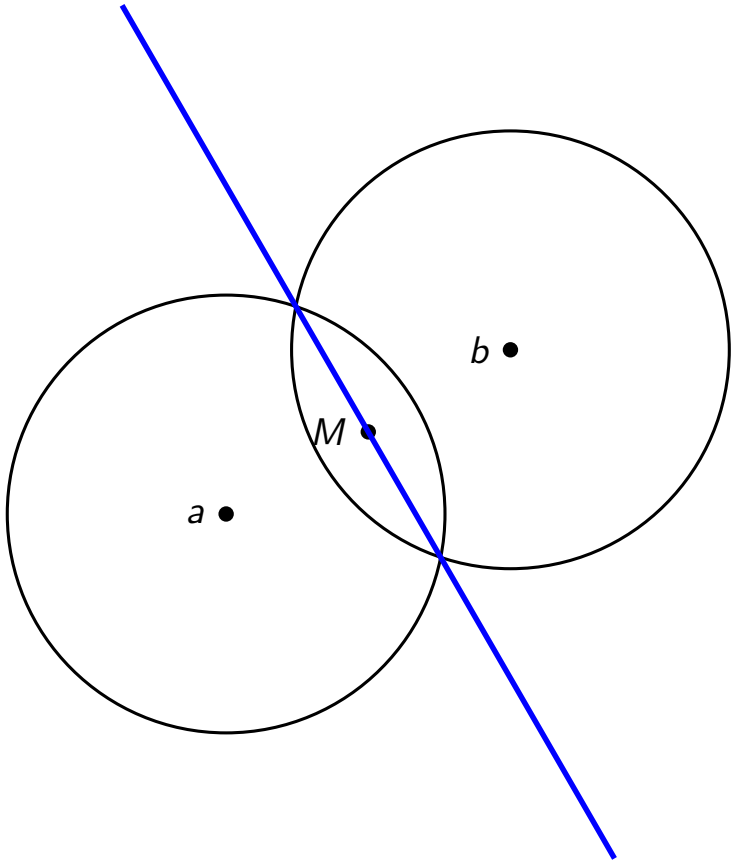
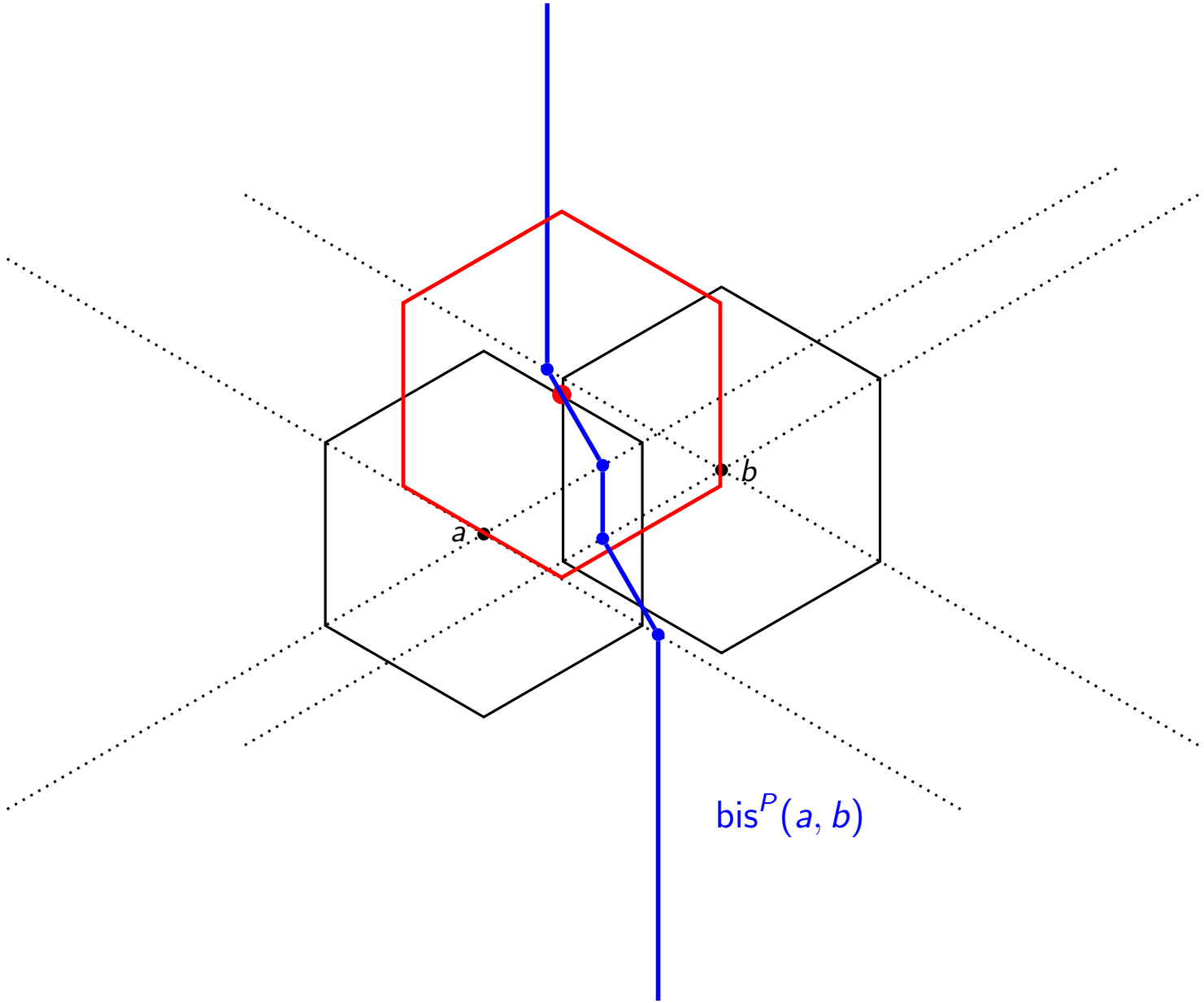
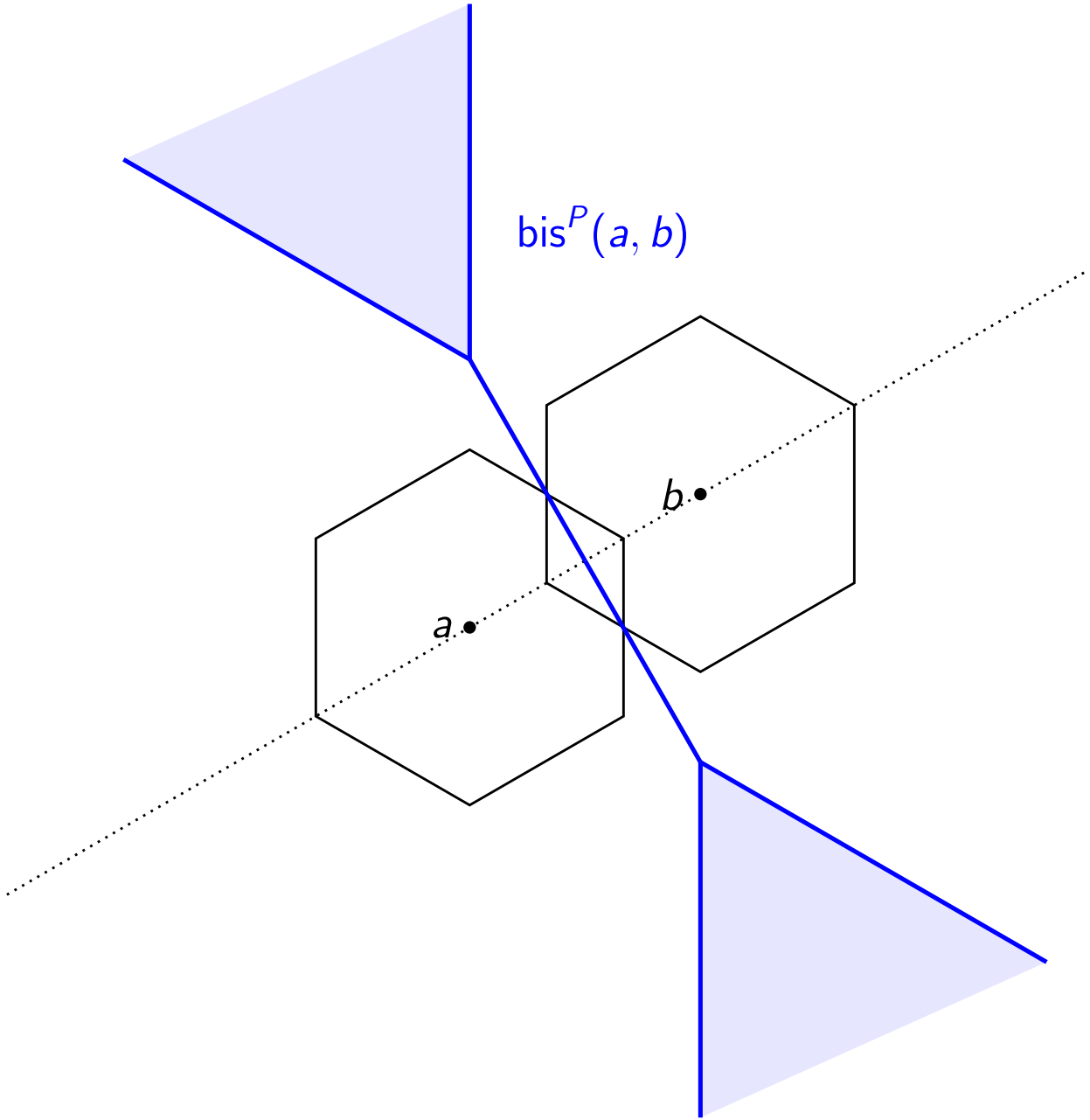


Figure: The Euclidean bisector of two points in the plane.

1-dimensional hexagonal bisector



2-dimensional hexagonal bisector



A brief history of bisectors

- Ma (2000): a characterisation of bisectors and Voronoi diagrams in the $d = 2, 3$ cases.
- Blömer and Kohn (2018): "the intersection of two bisectors in $\mathbb{R}^d$ is a $(d-2)$-manifold."
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Our aim:

Study further structural properties of the bisector - find a characterisation and complexity of non-empty maximal cells - and specialise to particular norms.

Bisectors in 2 dimensions

Ma (2000): For convex $P \subset \mathbb{R}^2$,

- If the line segment $[a, b]$ is not parallel to any line segment of ∂P, then the bisector $\text{bis}^P(a, b)$ is homeomorphic to a line.

- $\text{bis}^P(a, b)$ contains one (resp. two) unbounded regions if and only if $[a, b]$ is parallel to one (resp. two) line segments of ∂P.

Polyhedrality of bisectors

Big question

What are the geometric, combinatorial, metric and set-theoretic properties of bisectors?

Some recent answers

$P \subset \mathbb{R}^d$ is a centrally symmetric polytope with $0 \in \text{int}(P)$.

Proposition (Criado, Joswig, Santos 2021)

The set $\text{bis}^P(a, b)$ is a polyhedral complex with maximal cells of the form

$$\text{bis}^P_{F,G}(a, b) := \text{bis}^P(a, b) \cap (a + C_F) \cap (b + C_G)\,.$$

for all choices of facets F, G of P. (C_F is the cone over the facet F.)

Weak and strong general position

Strong general position: "$\text{bis}^P(a', b')$ has same polyhedral structure as $\text{bis}^P(a, b)$ for small perturbations (a', b') of (a, b)."

Weak general position: a, b do not lie on a hyperplane parallel to any facet of P.

Theorem (Criado, Joswig, Santos 2021)

If (a, b) is in weak general position, then $\text{bis}^P(a, b)$ is homeomorphic to $\mathbb{R}^{d-1}$.

Proposition (Criado, Joswig, Santos 2021)

If (a, b) is in strong general position, then $\text{bis}^P(a, b)$ is a pure polyhedral complex of dimension $d - 1$.

Maximal cells of bisectors

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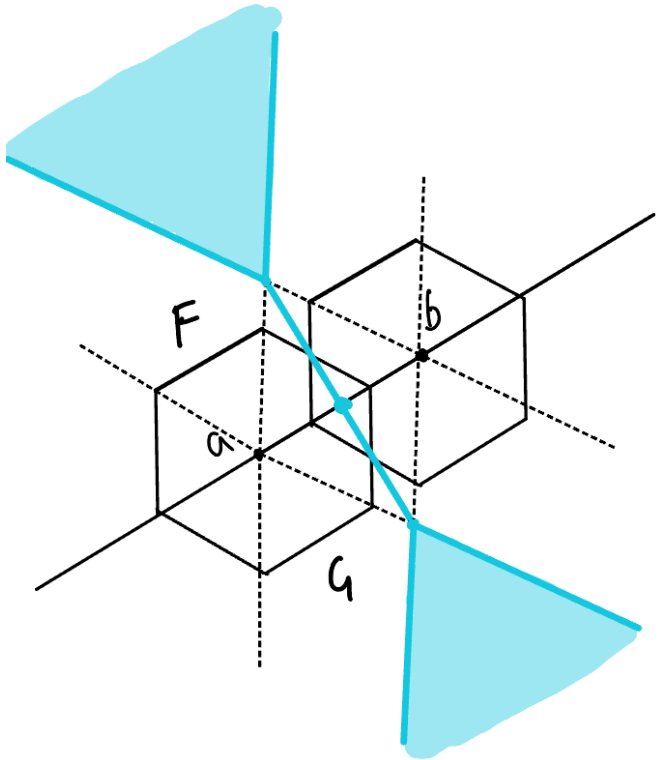


Figure: Non-pure 2-dimensional hexagonal bisector

Non-empty maximal cells of the bisector

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- Given facets F, G, we want to know when $\mathrm{bis}^P_{F,G}(0,a) \neq \emptyset$.

- Collect all such $a \in \mathbb{R}^d$ into a set called the *bisection cone* of F and G:

$$\mathcal{B}_{F,G} = \{a \in \mathbb{R}^d : \mathrm{bis}^P_{F,G}(0,a) \neq \emptyset\}.$$

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$$\mathcal{B}_{F,G} = \{a \in \mathbb{R}^d : \text{bis}^P_{F,G}(0,a) \neq \emptyset\}.$$
- $\mathcal{B}_{F,G}$ is indeed a polyhedral cone.

Bisection cones

$$\mathcal{B}_{F,G} = \{a \in \mathbb{R}^d : \mathrm{bis}^P_{F,G}(0, a) \neq \emptyset\}$$

Proposition (J., Jochemko, 2023+)

The bisection cone $\mathcal{B}_{F,G}$ has the following ray description:

$$\mathcal{B}_{F,G} = \mathrm{cone}(\{v - u : v \in \mathrm{vert}(F), u \in \mathrm{vert}(G)\}).$$

- $\mathcal{B}_{F,G}$ is a slice of the Minkowski sum of the homogenizations of F and $-G$.
- Facet description of $\mathcal{B}_{F,G}$ is involved, useful for computaional purposes.

Bisection cones

Corollary (J., Jochemko 2023+)

For any facet $F = \{x \in P : z^T x = b\}$ of P

$$\begin{aligned} \mathcal{B}_{F,-F} &= C_F \quad \text{(the cone over the facet } F\text{)} \\ \mathcal{B}_{F,F} &= \{x \in \mathbb{R}^d : z^T x = 0\}. \end{aligned}$$

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Proof.

Set $G = -F, F$ successively in

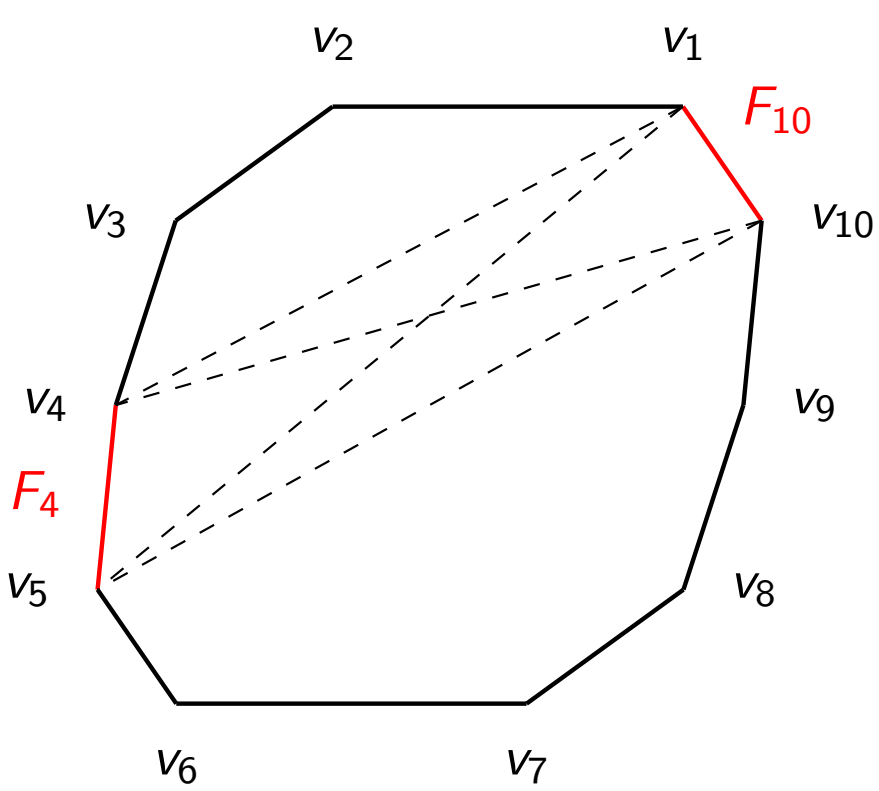
$$\mathcal{B}_{F,G} = \text{cone}(\{v - u : v \in \text{vert}(F), u \in \text{vert}(G)\}).$$

With $G = -F$, we get the conical hull of $\{a + b : a, b \in \text{vert}(F)\}$; this is C_F. With $G = F$, we get the conical hull of $\{a - b : a, b \in \text{vert}(F)\}$; this is the hyperplane parallel to F and passing through the origin. □

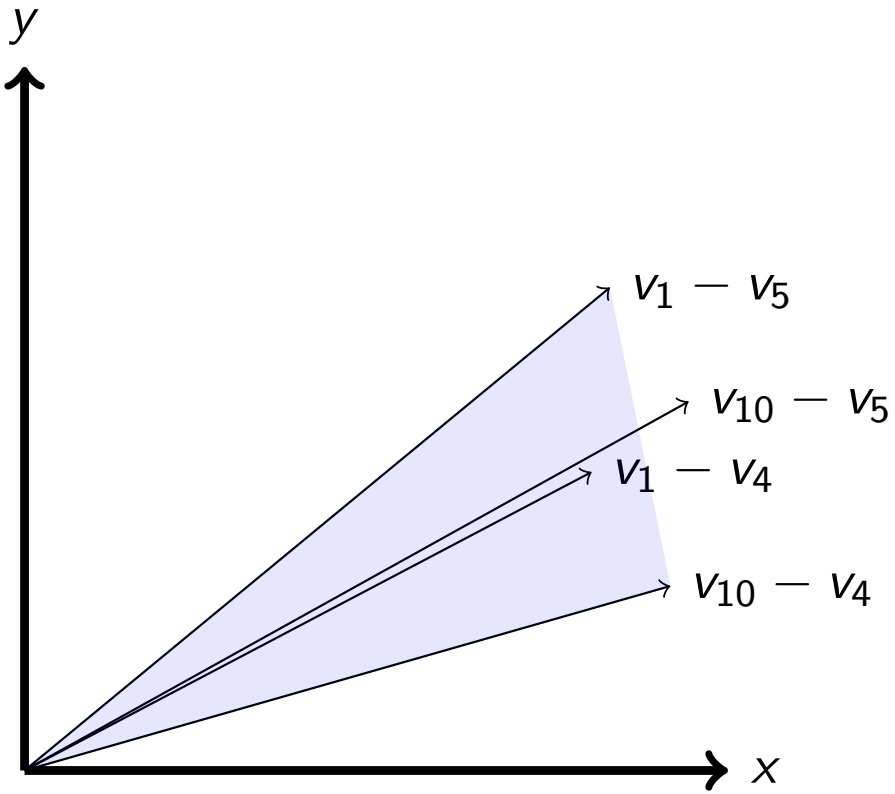
Bisection cones of polygons

$$\mathcal{B}_{F,G} = \text{cone}(\{v - u : v \in \text{vert}(F), u \in \text{vert}(G)\}).$$

Generators of $\mathcal{B}_{F,G}$ $\longleftrightarrow$ Diagonals of quadrilateral



(a) F_{10}, F_4 and relevant diagonals.



(b) Generators of $\mathcal{B}_{10,4}$.

Equivalence of bisectors

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- That is, for $a, b, a', b' \in \mathbb{R}^d$, declare $\text{bis}^P(a, b)$ to be *equivalent* to $\text{bis}^P(a', b')$ if

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  for all choices of facets F, G of P.
- Let $a \sim_P b$ if $\mathrm{bis}^P(0, a)$ is equivalent to $\mathrm{bis}^P(0, b)$.

Equivalence classes under $\sim_P$

- $a \sim_P b \iff$ for every pair of facets F, G of P, we have $a \in \mathcal{B}_{F,G}$ if and only if $b \in \mathcal{B}_{F,G}$.

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- Equivalence classes under $\sim_P$ are of the form

$$E_S = \bigcap_{(F,G)\in S} \mathcal{B}_{F,G} \cap \bigcap_{(F,G)\in S^c} \mathcal{B}^c_{F,G}$$

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- What are the polyhedral properties of E_S?

- When $P = \mathbb{B}^d$, the tropical unit ball, the collection $\{\overline{E_S}\}$ forms a polyhedral fan [Criado, Joswig, Santos 2021].

The bisection fan of a polytope

Definition (Criado, Joswig, Santos 2021)

Let P be a polytope having the origin in its interior and let Δ be a polyhedral fan. Then Δ is the **bisection fan** of P if and only if for all $a, b \in \mathbb{R}^d$ in general position the following two conditions are equivalent.

(i) The bisectors $\text{bis}^P(0, a)$ and $\text{bis}^P(0, b)$ are equivalent, i.e. $a \sim_P b$.

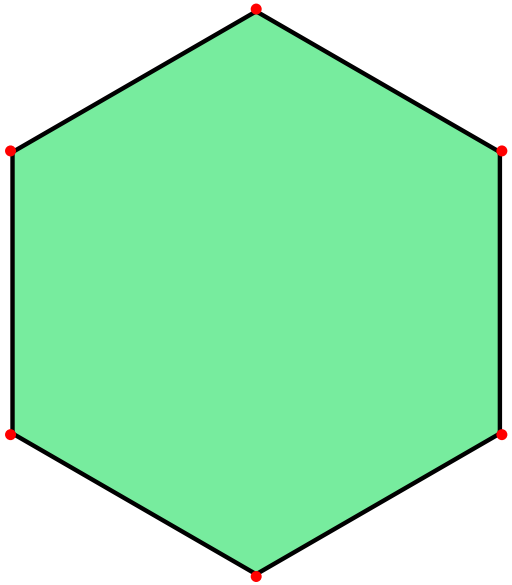
(ii) Both points a and b lie in the interior of the same maximal cone of Δ.

The *bisection fan* of a polytope, if it exists, captures the polyhedrality of the equivalence classes of points under $\sim_P$.

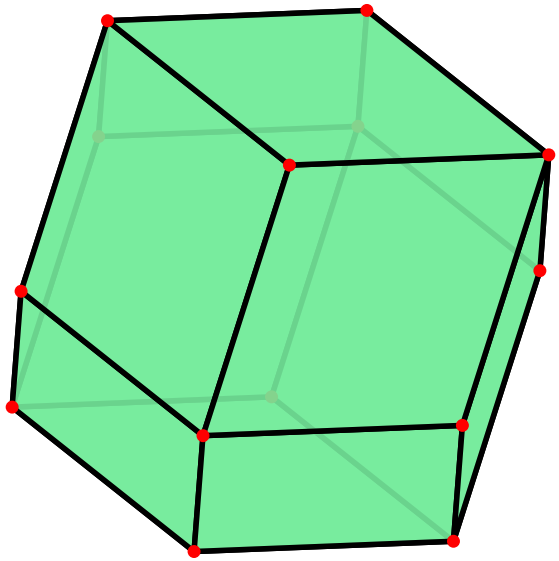
Tropical unit ball

The unit ball corresponding to the tropical norm is defined by

$$\mathbb{B}^d = \bigcap_{i \neq j} \{x \in \mathbb{R}^{d+1}/\mathbb{R}1 : x_i - x_j \leq 1\}$$



(a) $\mathbb{B}^2$ is a hexagon.



(b) $\mathbb{B}^3$ is a rhombic dodecahedron.

Existence of bisection fans

Theorem (Criado, Joswig, Santos 2021)

The bisection fan of the tropical unit ball $\mathbb{B}^d$ exists and is equal to the fan of *bisected ordered partitions* of $d+1$.

Theorem (J., Jochemko, 2023+)

The bisection fan of P exists when P is among the following cases:

1. Centrally symmetric polygons
2. Cross-polytopes
3. Hypercubes
4. Symmetric edge polytopes of K_d (the polar dual of $\mathbb{B}^d$.)

Bisection fan of polygons

Proposition (J., Jochemko, 2023+)

Let $P \subset \mathbb{R}^2$ be a centrally symmetric $2n$-gon with vertices $\{v_1, \ldots, v_{2n}\}$. The bisection fan of P is the complete polyhedral fan Δ the rays of which are given by $v_i - v_j$, for every $i, j \in [2n]$, $i \neq j$.

Bisection fan of a 10-gon

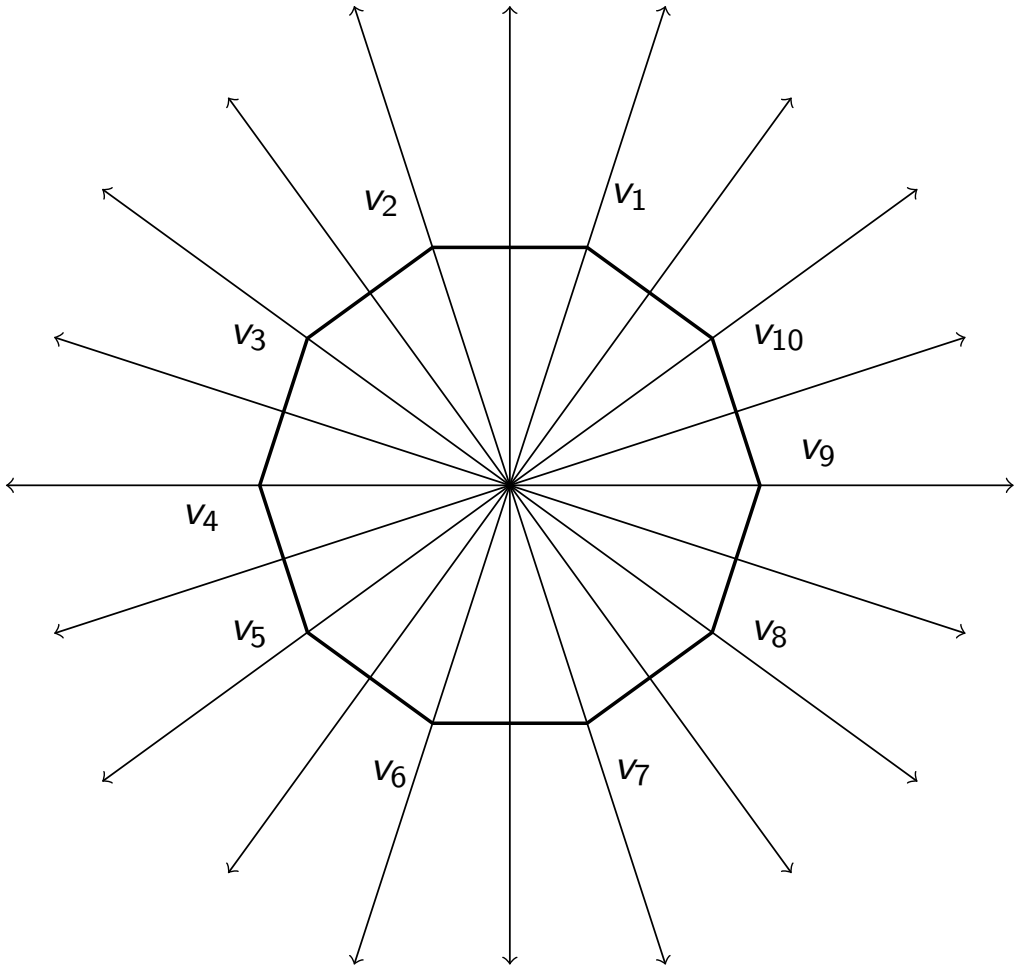


Figure: Bisection fan of a 10-gon with rays indicated

Bisection fan of cross-polytopes

- The d-dimensional cross-polytope is defined by

$$\diamond_d = \mathrm{conv}\{\pm e_i : i \in [d]\}$$

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Proposition (J., Jochemko, 2023+)

The bisection fan of $\diamond_d$ is the fan induced by the hyperplane arrangement

$$\mathcal{H} = \left\{ \{x \in \mathbb{R}^d : x_i = 0\}_{i=1,\ldots,d} \bigcup \left\{ x \in \mathbb{R}^d : \sum_{i \in I} x_i = \sum_{i \in I^c} x_i \right\}_{I \subseteq 2^{[d]}} \right\}$$

Bisection fan of cross-polytopes

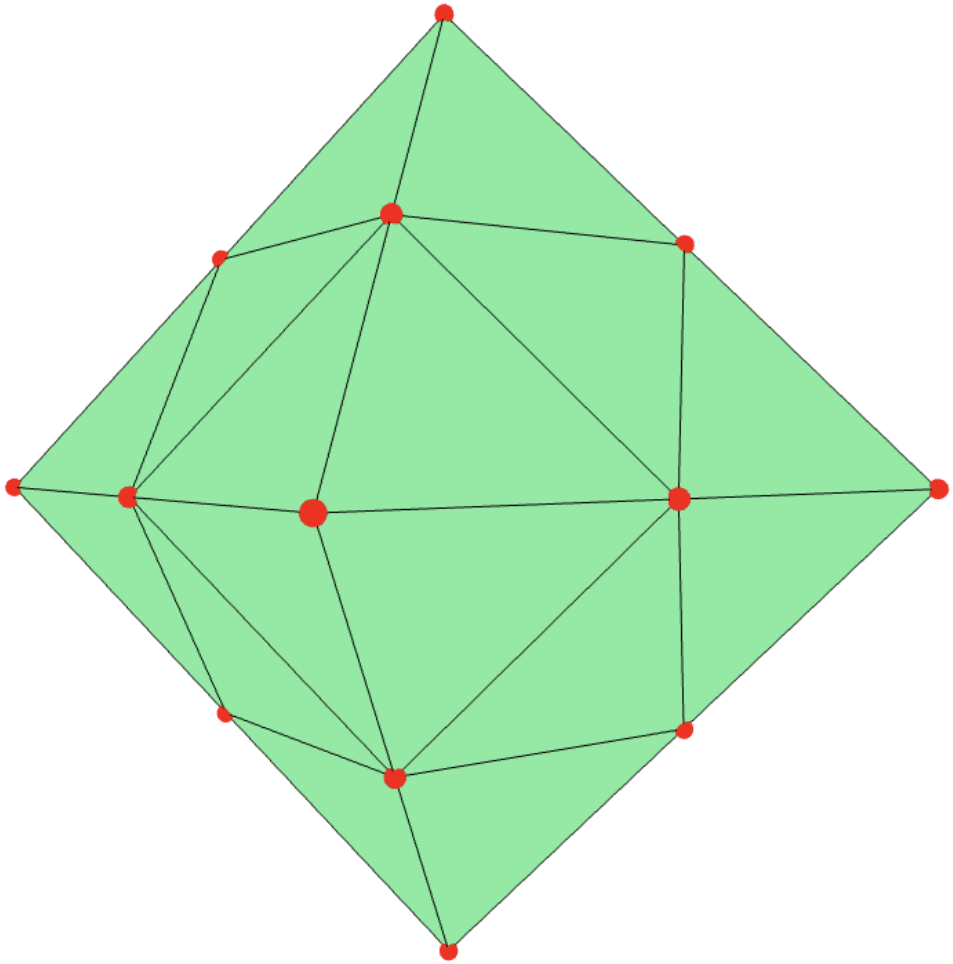


Figure: Bisection fan of the octahedron (intersected with octahedron)

Bisection fans wrt other fans

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- Not every bisection fan has this property: e.g. the cube.
- More complicated for the symmetric edge polytope of K_d.

Bisection fans wrt other fans

$\mathcal{F}_P =$ face fan of $P = \{\text{cone}(F) : F \text{ is a face of } P\}$,

$\mathcal{G}_P =$ hyperplane arrangement consisting of hyperplanes passing through the origin and parallel to a facet hyperplane,

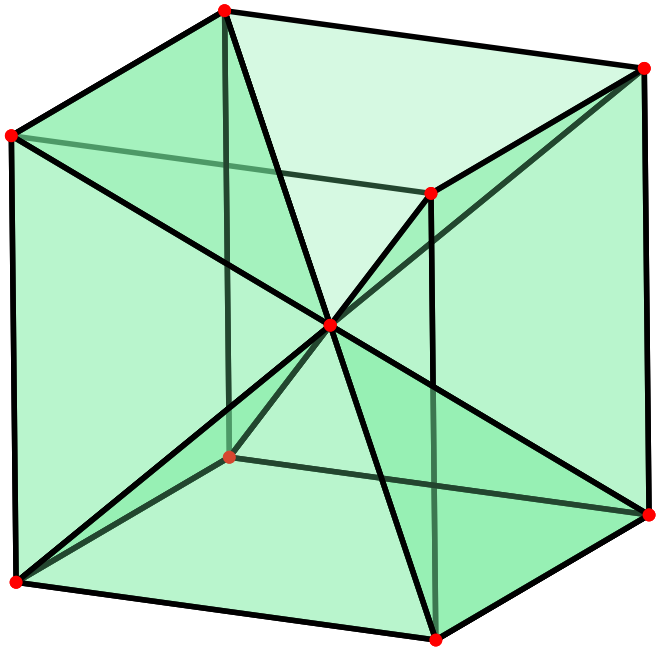
$\mathcal{H}_P =$ fan induced by $\mathcal{G}_P$.

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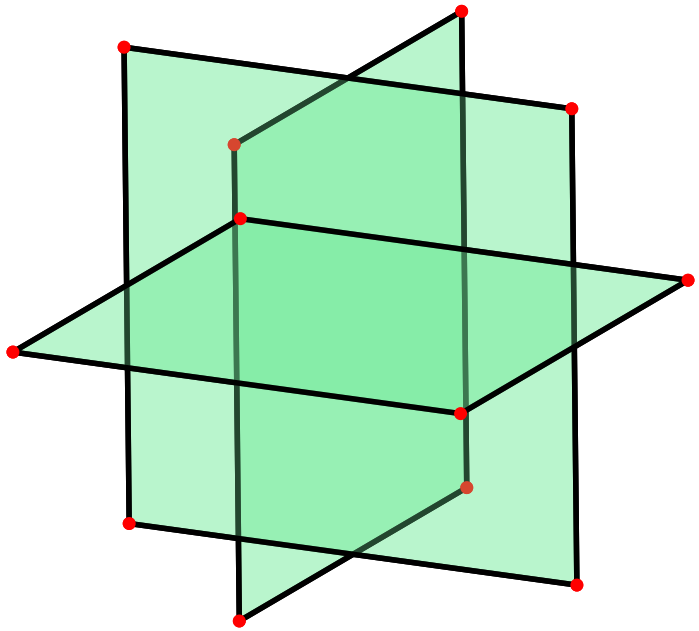
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(a) $\mathcal{F}_P$ when $P = [-1, 1]^3$



(b) $\mathcal{G}_P$ when $P = [-1, 1]^3$

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Proposition (J., Jochemko 2023+)

If a centrally symmetric polytope P admits a bisection fan Δ, then Δ refines both of $\mathcal{F}_P$ and $\mathcal{H}_P$.

Future directions

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